

# Numerical study of bubble dynamics with the Boundary Element Method

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## **Abstract.**

A new Boundary Element Method software for simulating bubble dynamics is presented. This tool allows us to simulate a wide variety of scenarios, including external acoustic fields and multiple bubble interaction. We present results on the interaction between two oscillating bubbles in various configurations. Different sized bubbles under the effects of acoustic fields at different frequencies have been considered.

## **1. Introduction**

The study of two-phase flow phenomena such as bubble dynamics is of significant interest for many terrestrial and space applications. In particular, the interaction between oscillating bubbles has attracted researchers attention since the early work of V. Bjerknes [1, 2, 3]. A good understanding of the interaction dynamics between two oscillating bubbles is crucial for the proper management of bubbly flows on ground and in space.

Two phase flow problems are an extension of the traditional fluid mechanics involving the dynamics of two different fluids in the same domain. The Navier Stokes equations combined with special considerations must be taken into account in order to describe the interaction between these two fluids. For a wide variety of problems it is convenient to simplify the model to the maximum that allows to describe the phenomenon with a reasonable accuracy. The Boundary Element Method (BEM) [4, 5, 6, 7, 8, 9, 10] provides a convenient, efficient way to describe two-phase flows in different scenarios, since it offers a low computational cost in comparison with other numerical methods, and provides an accurate representation of the interface.

In this paper we present a study on the interaction of two oscillating bubbles in different configurations by means of a BEM. We have considered bubbles of different sizes in an inviscid liquid under the effects of an external acoustic field at different frequencies. In Section 2 the BEM is introduced. The numerical procedure as well as the software package specially built for this work are presented in Section 3. The implemented model is validated by means of the comparison with previous numerical results in Section 4. In Section 5 we present results on the interaction between two oscillating bubbles in different scenarios.

## **2. Boundary Element Method**

The boundary element formulation for three dimensional fluid dynamics is commonly used for the simulation of bubble dynamics and other problems involving mechanical stress analysis.

The BEM has been used in the study of the dynamics of toroidal bubbles [5], and in the study of cavitation bubbles near a wall [10]. Many BEM formulations for bubble dynamics have been developed in the past years, including indirect methods [6] and completely desingularized methods [8]. In the BEM, the phenomenon of study is well defined over a domain with a set of initial and boundary conditions. This method has some limitations in fluid mechanics, since only potential flow models accomplish its requirements. However, three dimensional simulations by means of the boundary element formulation are faster than simulations with other numerical methods such as finite elements. This allows the BEM to capture more detailed information about the physical boundary than other methods with the same computational cost. In this work we have considered a BEM with inviscid liquid.

An incompressible and irrotational flow can be defined by the Laplace equation  $\nabla^2\phi = 0$ ,  $\phi$  being a potential defined over the fluid domain. Let us consider a bubble in an unbounded domain  $\Omega$  governed by this potential. The velocity of the bubble surface can be determined from:

$$\frac{d\vec{x}}{dt} = \nabla\phi \quad (1)$$

In order to completely define the dynamics of the bubble, it is required to obtain a function that satisfies the governing equation, and to derive a set of boundary conditions for the bubble interface. The Green function  $G$  is one solution of the Laplace equation in the three dimensional domain:

$$G = \frac{1}{|\vec{P} - \vec{Q}|} \quad (2)$$

Eq. 2 relates the potential in  $\vec{P}$  and  $\vec{Q}$  due to a source in  $\vec{Q}$  with potential of unitary value. Defining  $\vec{r} = \vec{P} - \vec{Q}$ , the velocity is obtained from Eq. 1:

$$\vec{v} = \vec{\nabla}G = -\frac{\vec{r}}{|\vec{r}|^3} \quad (3)$$

In order to complete the definition of the problem, the time dependence and the conservation of momentum are introduced as boundary conditions. For the velocity, the time dependence is introduced through the kinematic boundary condition in the bubble surface:

$$\frac{d\vec{x}}{dt} \cdot \hat{n} = \nabla\phi \cdot \hat{n}, \quad (4)$$

where  $\hat{n}$  is the normal to the surface. Thus, normal velocities in the surface of the bubble must be continuous.

The momentum conservation is derived through the Bernoulli equation, relating the physical quantities at point of the fluid with the values at the far field:

$$\frac{p_\infty}{\rho} + \frac{1}{2}|\vec{v}_\infty|^2 + \frac{\partial\phi_\infty}{\partial t} - E_\infty = \frac{p}{\rho} + \frac{1}{2}|\vec{v}|^2 + \frac{\partial\phi}{\partial t} - E \quad (5)$$

where  $p$  is the liquid pressure,  $p_\infty(t)$  is the pressure at the far field,  $\rho$  is density, and  $E$  is the potential energy of external sources such as gravity. If the far field is at rest, and the potential origin is taken at infinity, the above equation reduces to:

$$\frac{\partial\phi}{\partial t} = -\frac{p_\infty - p}{\rho} - \frac{1}{2}|\vec{v}|^2 + (E - E_\infty), \quad (6)$$

which can also be expressed in terms of the material derivative of the potential:

$$\frac{D\phi}{Dt} = \frac{\partial\phi}{\partial t} + \vec{v}\nabla\phi = \frac{1}{2}|\vec{v}|^2 - \frac{p_\infty - p}{\rho} + (E - E_\infty) \quad (7)$$

In order to determine the time evolution of  $\phi$ , the velocity of the bubble surface must be provided. The BEM calculates the velocity from the boundary integral equation given a set of initial conditions for the velocity potential:

$$\frac{1}{2}\phi(\vec{P}) = \int_{\partial\Omega} \left( \frac{\partial\phi}{\partial n}(\vec{Q})G(\vec{P}, \vec{Q}) - \phi(\vec{Q})\frac{\partial G}{\partial n}(\vec{P}, \vec{Q}) \right) d\Omega_Q \quad (8)$$

### 3. Numerical Procedure

The numerical procedure followed to implement the model is based on [6, 9]. In order to solve the equations, the interface is discretized in  $n$  triangular elements, while the potential and velocity are interpolated using linear basis functions. The discretized form of Eq. 8 is given by:

$$\frac{1}{2}\phi(\vec{P}_i) = \sum_{j=0}^n \left( \int_{s_j} \left( \frac{\partial\phi}{\partial n}(\vec{Q})G(\vec{P}_i, \vec{Q}) - \phi(\vec{Q})\frac{\partial G}{\partial n}(\vec{P}_i, \vec{Q}) \right) ds_j \right) \quad (9)$$

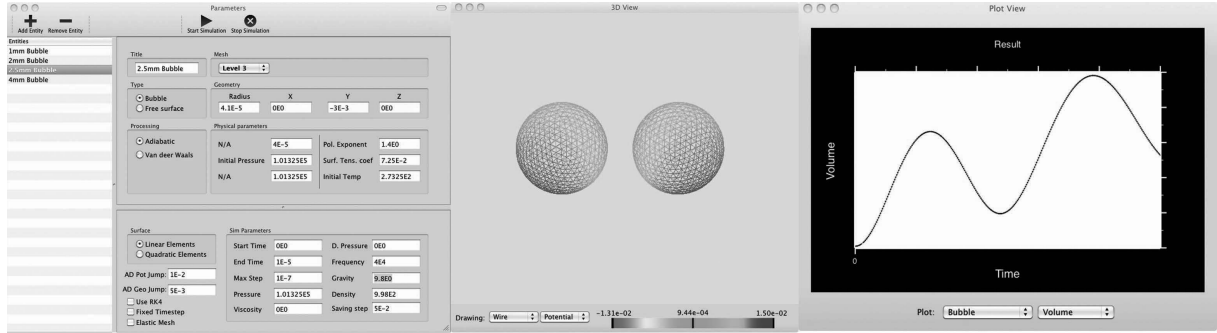
Eq. 9 is integrated over all the surface patches  $s$  in order to form a linear set of equations to obtain the normal velocities  $\frac{\partial\phi}{\partial n}$ . We used the Bi-Conjugate Gradient Stabilized solver (Bi-CG Stab) in our code. The tangential component of the velocity is obtained by finite differences from the values of the potential at a given point. The control points are then advanced in time by means of the discretized dynamic and kinematic boundary conditions:

$$\frac{D\phi_i}{Dt} = \frac{1}{2}|\vec{v}_i|^2 - \frac{p_\infty(t) \left( 1 - \left( \frac{V_0}{V} \right)^\gamma + 2\sigma\kappa_i \right)}{\rho} - g \cdot h_i \quad (10)$$

$$\frac{D\mathbf{x}_i}{Dt} = \nabla\phi_i \quad (11)$$

where  $V_0$  is the initial volume of the bubble,  $\sigma$  is the surface tension,  $\kappa_i$  is the principal curvature of the liquid-air interface,  $g$  is gravity, and  $h_i$  is the height of a bubble point measured from the inception point. The control points are advanced with its material velocity in order to avoid interpolation of the values for the potential at the next timestep. Numerical instabilities in our method are avoided by imposing a stability criteria limiting both the time step and the maximum displacement of a given node. Both a fourth order Runge-Kutta and a predictor-corrector schemes are implemented in the code.

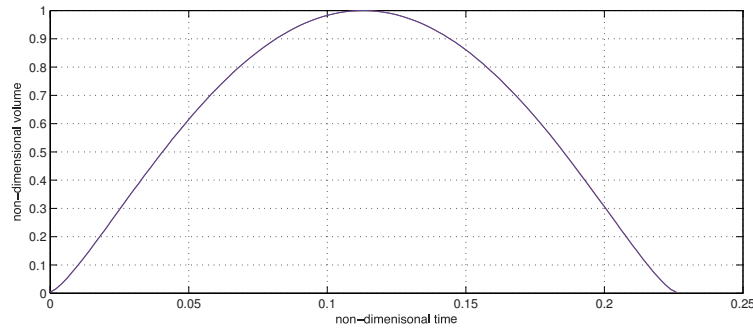
Due to the intensive calculations that have to be carried out in the BEM, interpreted languages such as MATLAB or MATHEMATICA were discarded. Instead, we constructed a C code that takes full advantage of multiple CPUs, which accelerates the simulations. The designed software integrates all the calculus in one package, with the ability to export the results to other softwares such as EXCEL or MATLAB. This software was developed for the Mac OS X platform, which provides a native set of libraries (Cocoa Framework) that allowed us to build a graphical user interface, shown in Fig. 1, without much overhead. The software also benefits from parallelization with multiple CPUs via Grand Central Dispatch and Hardware acceleration of the 3D viewport via OpenGL framework. On the Windows side, the Graphical User Interface was designed with the Cocotron libraries, while the parallelization was carried out with openMP.



**Figure 1.** Main interface of the software.

#### 4. Validation

In order to validate our BEM code, some tests have been carried out to compare results provided by the BEM simulations with existing results. We initially tested the code by simulating an oscillating air bubble in water with  $R_0 = 0.1651$  and  $\epsilon = 100$ , where  $R_0$  and  $\epsilon$  are the dimensionless initial radius and pressure, respectively. This configuration corresponds to an overpressure of the bubble which generates its volume oscillation and shows a highly varying dynamics. Thus, it is an optimum phenomenon to validate the BEM. Fig. 2 shows the evolution of the bubble volume with time. The behaviour presented in Fig. 2 is in complete agreement with the results obtained by means of the Rayleigh-Plesset equation.

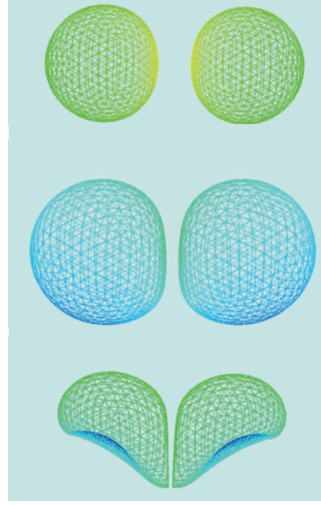


**Figure 2.** Time evolution of the volume of an underwater explosion bubble obtained by means of the BEM.

Our BEM code has also been validated by the comparison with results from previous studies with the BEM. Fig. 3 shows the time evolution of two in-phase bubbles under the action of gravity, as one can encounter when bubbles are generated by two underwater exploding charges. No remeshing or smoothing was required in order to advance the bubble surface in time. The behaviour of the interacting bubbles shown in Fig. 3 is in good agreement with previous results with a BEM (see Fig. 6 in [7]). Thus, our code is proved to be an appropriate tool to reproduce scenarios with multiple bubbles.

#### 5. Bubble interactions

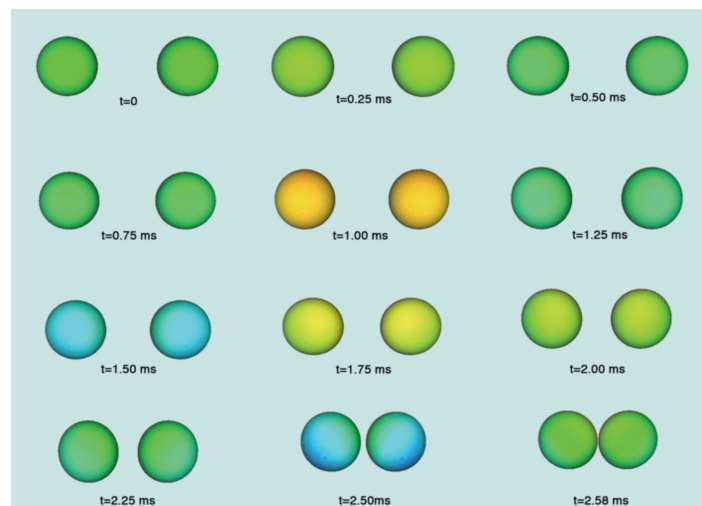
We have analyzed the interaction of two bubbles under the effects of an external acoustic field with no spatial dependence. Although finite wavelength standing wave conditions have also been implemented in the code, we do not use them in this work since we aim to isolate as much as



**Figure 3.** Time evolution of two in-phase bubbles obtained by means of the BEM.

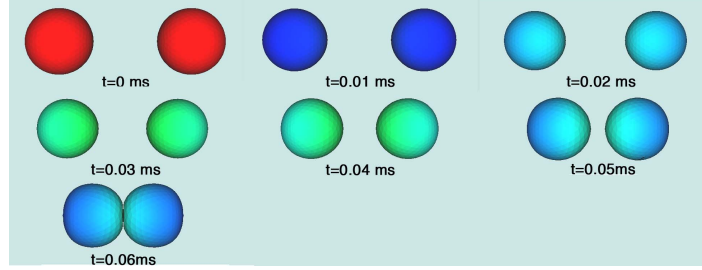
possible the effects of the acoustic field generated by the pulsating bubbles (secondary Bjerkens force). Had we also considered standing waves of finite wavelength, additional forces would act on the bubbles. Thus, the following results show the behaviour of bubbles oscillating under the effects of an external acoustic field without spatial dependence and a spatially dependent acoustic field generated by the pulsation of the neighboring bubble.

We have considered bubbles of different size and different frequencies of the external acoustic field. Fig. 4 shows the time evolution of two bubbles with an initial  $0.1\text{mm}$  radius at an initial separation between their geometric centers of  $0.4\text{mm}$  under an acoustic field at  $5\text{ KHz}$ . As a result of the attractive second Bjerknes force, each bubble approaches the other until they collide after  $2.58\text{ms}$ . An increasing attraction velocity with decreasing distance between bubbles can be observed.



**Figure 4.** Time evolution of two  $0.1\text{mm}$  bubbles initially separated  $0.4\text{mm}$  under the presence of an external acoustic field at  $5\text{ KHz}$ .

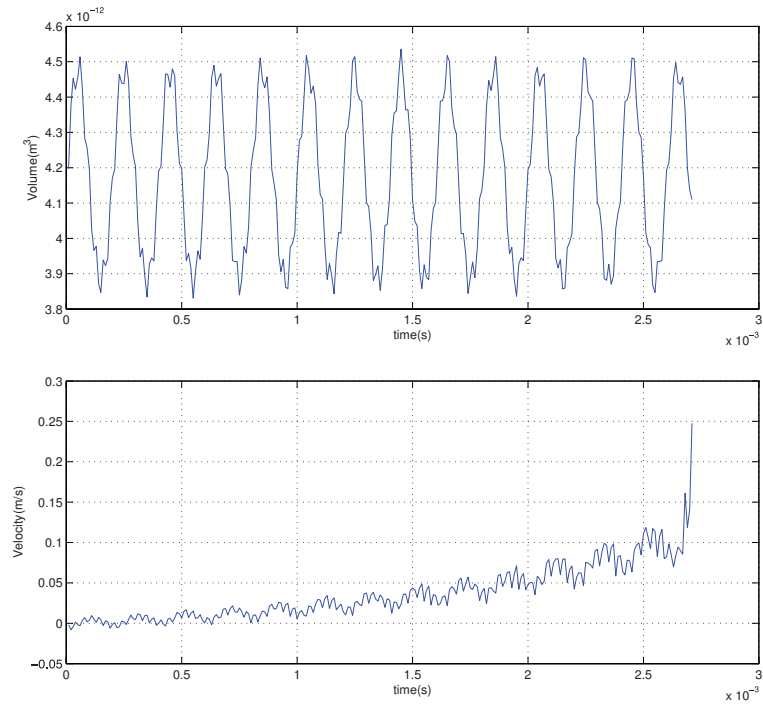
A similar behaviour is obtained in Fig. 5, where two bubbles of  $25\mu m$  evolve under the effects of an external acoustic field with frequency 57 MHz from an initial separation distance of  $0.1mm$ . In this case, the deformation of both bubbles when they are very close can also be observed. One would expect that when the liquid film between the bubbles was expelled, coalescence would take place. However, the coalescence phenomenon is not included in the current version of our BEM code. Thus, additional boundary conditions should be implemented in the BEM in order to simulate coalescence.



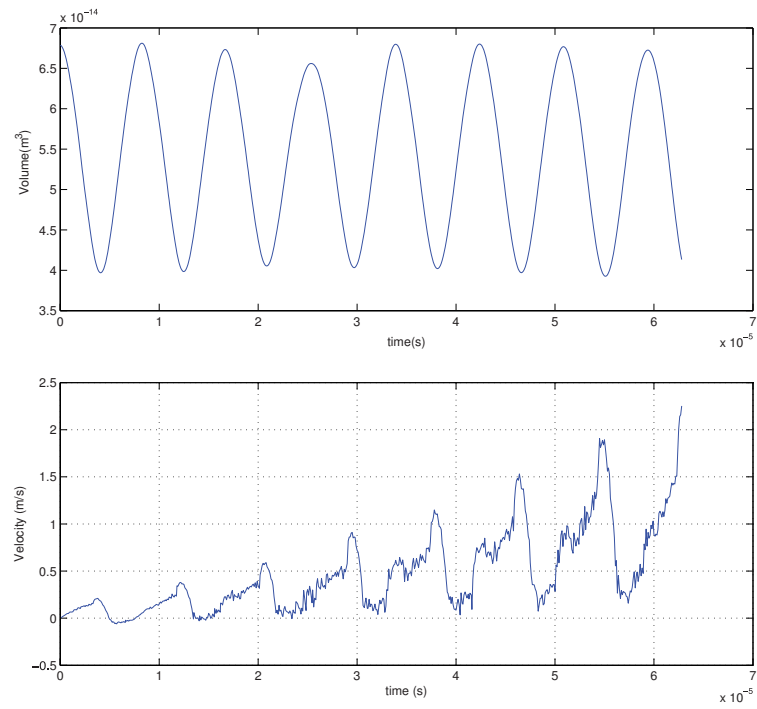
**Figure 5.** Time evolution of two  $25\mu m$  bubbles initially separated  $0.1mm$  under the presence of an external acoustic field at 57MHz.

The time evolution of the volume of one bubble as well as the velocity of the geometric center of the bubble is shown in Fig. 6, where the parameters are the same as in Fig. 4. On the one hand, the velocity is found to increase with time, as bubbles approach. On the other hand, oscillations in both volume and velocity can be observed. The bubble velocity shows two frequencies of oscillation. The lowest frequency coincides with the volume oscillation frequency, while the highest frequency of oscillation of the velocity is related to the acoustic field oscillations.

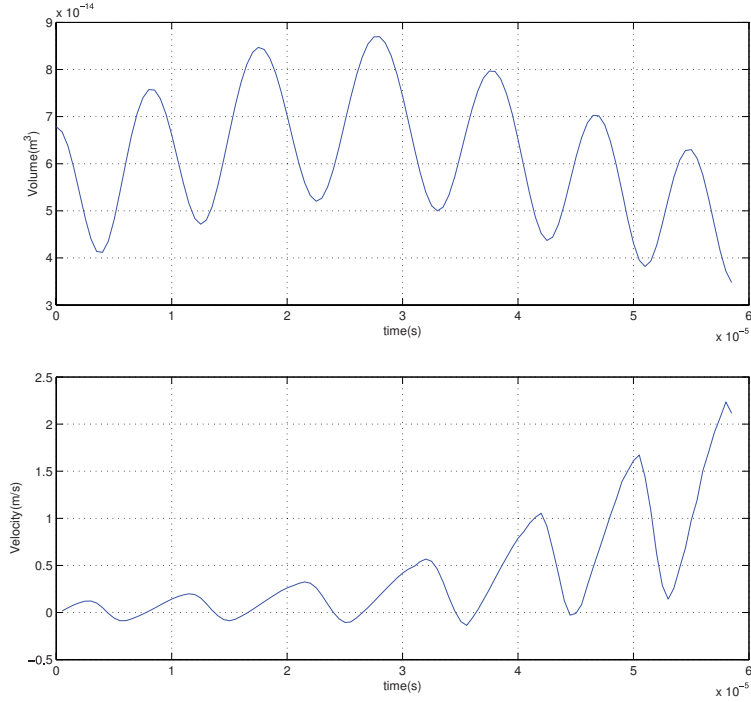
A similar oscillation pattern than the one shown in Fig. 6 is obtained with two bubbles of  $25\mu m$  initially at a distance of  $0.1mm$  under an acoustic field at 57 MHz (Fig. 7). The smallest frequency of the velocity coincides with the frequency of the volume oscillations, while the largest velocity frequency is related to the external field frequency. However, this behaviour is not observed in Fig. 8, where the volume and velocity time evolutions of a  $25\mu m$  bubble initially at  $0.1mm$  from another same sized bubble under an acoustic field at 5 KHz are presented. This behavior is not observed when the bubble collapse time is comparable with the period of oscillation, as shown in Fig. 8. In this case, the attractive velocity seems to be only modulated by the effects of the volume oscillations.



**Figure 6.** Time evolution of the volume and velocity of a bubble with parameters as in Fig. 4.



**Figure 7.** Time evolution of the volume and velocity of a bubble with parameters as in Fig. 5.



**Figure 8.** Time evolution of the volume and velocity of a  $25\mu\text{m}$  bubble under an acoustic field at 5 KHz.

The volume oscillations in all three configurations suggest that the attraction velocity decreases when the bubble is expanding. This can be explained from the fact that the expanding bubble needs to push away the surrounding fluid, which slows down the attraction velocity of the bubbles. On the other hand, the collapse phase of the bubble accelerates the attraction as the pressure of the exterior sides of the bubble decreases and allows the bubble to advance more rapidly.

Finally, by comparing velocities in Figs. 6-8, one can observe that the attraction velocity is larger for smaller sized bubbles. This observation is in agreement with results on particle attraction.

## 6. Conclusions

We have presented a new embedded numerical package that allows us to study a wide range of bubble dynamics scenarios that can be defined by a BEM. This software is designed with the aim to have the maximum performance on modern computers with multiple CPUs. The software has been validated with existing numerical results. We have presented results on the interaction between two oscillating bubbles in various configurations. The effects of the secondary Bjerknes force have been analyzed.

## 7. Acknowledgements

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